**Utility Curves**

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| **Utility Curves** |
| Reference Lotteries  Utility Functions  Empirical Data |

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| **Reference Lotteries** | | |
| To determine the utility curve using reference lotteries is merely the process of posing multiple lotteries to the decision maker and eliciting the certainty equivalent, CE, for each risky decision. Then plot values to form the utility curve. | | |
|  | 50-50 Reference Lotteries. This classic approach is to construct a series of reference lotteries containing equal chance probabilities and elicit the CE for different outcome values. |  |
|  | Probability Equivalent, PE, Reference Lotteries. Another approach is to construct a series of reference lotteries containing CE=0 and elicit probabilities that result in an indifference lottery for different outcome values. |  |

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| **Utility Functions** | | |
| A function is chosen to represent the utility curve and the parameters are chosen to reflect the decision maker’s attitude toward risk. Let X=evaluation measure and U=utility measure. Then possible utility functions are: | | |
|  | U=1-exp(-X/R) , R=Risk Tolerance |  |
|  | U=Ln(X) |  |
|  | U=X^(0.5) |  |
|  | U=X/(X+K) , K=constant |  |

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| **Empirical Data** |
| Estimate a utility curve from observed behavior. A procedure will be presented but mostly the intent of this section is to represent a large class of approaches that will emerge from the logic of utility theory through observing the data. |

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|  | **Decision Analysis with Utility Curves** |  |
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| *Example* | | |
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|  | **Utility Curves: Reference Lotteries** |  |

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| **Reference Lotteries** | | |
| To determine the utility curve using reference lotteries is merely the process of posing multiple lotteries to the decision maker and eliciting the certainty equivalent, CE, for each risky decision. Then plot values to form the utility curve. | | |
|  | 50-50 Reference Lotteries. This classic approach is to construct a series of reference lotteries containing equal chance probabilities and elicit the CE (Certainty Equivalent) for different outcome values. |  |
|  | Probability Equivalent, PE, Reference Lotteries. Another approach is to construct a series of reference lotteries containing CE=0 and elicit probabilities that result in an indifference lottery for different outcome values. |  |

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| **Boundaries** | **A-0** |  |  |  |  |  |  |
|  |  |  |  | (1.0) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.0) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **+$10** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | For CE=+$10 and P(Win)=1, set U(+$10)=1 | | | | | | |

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|  | **A-0** |  |  |  |  |  |  |
|  |  |  |  | (0.0) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (1.0) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **–$10** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | For CE= **–**$10 and P(Win)=0, let U(**–**$10)=0 | | | | | | |

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| **Boundaries** |  | |  |
|  | From Lotteries | |  |
|  | $ | Utiles |  |
|  | +10 | 1.0 |  |
|  | –10 | 0.0 |  |
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| **Certainty Equivalent, CE, 50-50 Reference Lotteries** | | |
|  | 50-50 Reference Lotteries. This classic approach is to construct a series of reference lotteries containing equal chance probabilities and elicit the CE (Certainty Equivalent) for different outcome values. |  |

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|  | **Boundaries** | |  |  | **From Lotteries** | |  |  |  |  | **Summary** | | | | |  |
|  | $ | Utiles |  |  | $ | Utiles |  |  |  |  | 50-50 | $ | U | CE | U |  |
|  | +10 | 1.0 |  |  | -10 | 0 |  |  |  | A-1 | 0.5 | 10 | 1 | -4 | 0.5 |  |
|  | –10 | 0.0 |  |  | -4 | 0.5 |  |  |  |  | 0.5 | -10 | 0 |  |  |  |
|  |  |  |  |  | -2 | 0.625 |  |  |  | A-2 | 0.5 | 10 | 1 | 1 | 0.75 |  |
|  |  |  |  |  | 1 | 0.75 |  |  |  |  | 0.5 | -4 | 0.5 |  |  |  |
|  |  |  |  |  | 10 | 1 |  |  |  | A-3 | 0.5 | 1 | 0.75 | -2 | 0.625 |  |
|  |  |  |  |  |  |  |  |  |  |  | 0.5 | -4 | 0.5 |  |  |  |
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| **Indifference** | **A-1** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.5) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.5) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **–$4** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**–4)=0.5U(+10)+0.5U(–10)=0.5(1)+0.5(0)=0.5** | | | | | | |

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| **Indifference** | **A-2** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.5) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.5) | Lose | **–$4** |  |
|  |  | NOGO |  |  |  | **+$1** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**+1)=0.5U(+10)+0.5U(–4)=0.5(1)+0.5(0.5)=0.75** | | | | | | |

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| **Indifference** | **A-3** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.5) | Win | **+$1** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.5) | Lose | **–$4** |  |
|  |  | NOGO |  |  |  | **–$2** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**–2)=0.5U(+1)+0.5U(–4)=0.5(0.75)+0.5(0.5)=0.625** | | | | | | |

This procedure can be continued until a curve is sufficiently defined.

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| **Probability Equivalent, PE, Reference Lotteries** | | |
|  | Probability Equivalent, PE, Reference Lotteries. Another approach is to construct a series of reference lotteries containing CE=0 and elicit probabilities that result in an indifference lottery for different outcome values. |  |

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|  | **Boundaries** | |  |  | **From Lotteries** | |  |  |  |  | **Summary** | | | | |  |
|  | $ | Utiles |  |  | $ | Utiles |  |  |  |  | PE | $ | U | CE | U |  |
|  | +10 | 1.0 |  |  | -10 | 0 |  |  |  | B-1 | 0.5 | 10 | 1 | -4 | 0.5 |  |
|  | –10 | 0.0 |  |  | -4 | 0.5 |  |  |  |  | 0.5 | -10 | 0 |  |  |  |
|  |  |  |  |  | -2 | 0.625 |  |  |  | B-2 | 0.75 | 10 | 1 | 1 | 0.75 |  |
|  |  |  |  |  | 1 | 0.75 |  |  |  |  | 0.25 | -10 | 0 |  |  |  |
|  |  |  |  |  | 10 | 1 |  |  |  | B-3 | 0.625 | 10 | 1 | -2 | 0.625 |  |
|  |  |  |  |  |  |  |  |  |  |  | 0.375 | -10 | 0 |  |  |  |
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| **Indifference** | **B-1** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.5) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.5) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **–$4** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**–4)=0.5U(+10)+0.5U(–10)=0.5(1)+0.5(0)=0.5** | | | | | | |

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| **Indifference** | **B-2** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.75) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.25) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **+$1** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**+1)=0.75U(+10)+0.25U(–10)=0.75(1)+0.25(0)=0.75** | | | | | | |

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| **Indifference** | **B-3** |  |  |  |  |  |  |
| **Lottery** |  |  |  | (0.625) | Win | **+$10** |  |
|  |  | GO |  |  |  |  |  |
|  |  |  |  | (0.375) | Lose | **–$10** |  |
|  |  | NOGO |  |  |  | **–$2** | 🡨CE |
|  |  |  |  |  |  |  |  |
|  | U(**–2)=0.625U(+10)+0.375U(–10)=0.625(1)+0.375(0)=0.625** | | | | | | |

This procedure can be continued until a curve is sufficiently defined.

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|  | **Decision Analysis with Utility Curves** |  |
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| *Example* | | |
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|  | **From Lotteries** | |  | **From Utility Curve** | |  |
|  | $ | Utiles |  | $ | Utiles |  |
|  | -10 | 0 |  | 6 | 0.8888889 |  |
|  | -4 | 0.5 |  | 2 | 0.7777778 |  |
|  | -2 | 0.625 |  | 0 | 0.7083333 |  |
|  | 1 | 0.75 |  | -5 | 0.4166667 |  |
|  | 10 | 1 |  |  |  |  |
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|  | $ | State1 | State2 | State3 | State4 | EMV | Decision |  |
|  | Decision A | 6 | 2 | 0 | -5 | 0.75 | Yes |  |
|  | Decision B | 0 | 0 | 0 | 0 | 0 |  |  |
|  | Probability | 0.25 | 0.25 | 0.25 | 0.25 |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  | Utiles | State1 | State2 | State3 | State4 | E[U] | Decision |  |
|  | Decision A | 0.8888889 | 0.7777778 | 0.7083333 | 0.4166667 | 0.6979167 |  |  |
|  | Decision B | 0.7083333 | 0.7083333 | 0.7083333 | 0.7083333 | 0.7083333 | Yes |  |
|  | Probability | 0.25 | 0.25 | 0.25 | 0.25 |  |  |  |
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|  | **Utility Curves: Utility Functions** |  |

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| **Utility Functions** | | |
| A function is chosen to represent the utility curve and the parameters are chosen to reflect the decision maker’s attitude toward risk. Let X=evaluation measure and U=utility measure. Then possible utility functions are: | | |
|  | U=1-exp(-X/R) , R=Risk Tolerance |  |
|  | U=log(X) |  |
|  | U=X^(0.5) |  |
|  | U=X/(X+K) , K=constant |  |

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|  | **Utility Curves: Utility Functions** |  |

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|  | **U=1-exp(-X/R) , R=Risk Tolerance** |  |
|  | **Example 1** |  |
| Find $Y such that the decision maker is Indifference between the GO and NOGO decisions. Then, R≈Y.   |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | **Indifference** | **C-0** |  |  |  |  |  |  | | **Lottery** |  |  |  | (0.5) | Win | **$Y** |  | |  |  | GO |  |  |  |  |  | |  |  |  |  | (0.5) | Lose | **–$Y/2** |  | |  |  | NOGO |  |  |  | **$0** | 🡨CE | |  |  |  |  |  |  |  |  | |  | U(**0)=0.5U(Y)+0.5U(–Y/2)** | | | | | | |   . . . | | |
| Consider the Reference Lottery.   |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | **Indifference** | **C-1** |  |  |  |  |  |  | | **Lottery** |  |  |  | (0.5) | Win | **+$4** |  | |  |  | GO |  |  |  |  |  | |  |  |  |  | (0.5) | Lose | **–$2** |  | |  |  | NOGO |  |  |  | **$0** | 🡨CE | |  |  |  |  |  |  |  |  | |  | **R≈4 and U($)=1** – **exp(** –**$/4)** | | | | | | |   . . . | | |
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|  | **Utility Curves: Utility Functions** |  |

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|  | **U=1-exp(-X/R) , R=Risk Tolerance** |  |
|  | **Example 2** |  |
| If U($) = 1–exp(–$/R), then $ = (–R)\*Ln(1–U($) ). Now, “$” is linear with “Ln(1–U($) )”  From the Reference Lotteries,   |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | |  |  |  |  |  |  |  | From the relationship, $ = (–R)\*Ln(1–U($) )  Regress “Ln(1–U($) )” on “$”  Regression yields, Slope= –7.9455 ≈ –8  Thus, R≈8 | |  | **From Lotteries** | |  |  | |  | |  | $ | Utiles |  | Ln(1 – U) |  |  | |  | -10 | 0 |  | 0 |  |  | |  | -4 | 0.5 |  | -0.69315 |  |  | |  | -2 | 0.625 |  | -0.98083 |  |  | |  | 1 | 0.75 |  | -1.38629 |  |  | |  | 10 | 1 |  |  |  |  | |  |  |  |  |  |  |  |   . . . | | |
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|  | **Decision Analysis with Exponential Utility Functions** |  |

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|  |  | Exponential Utility Curve  U($)=1–exp(–$/R) | |  |
|  | $ | Utiles, R=4 | Utiles, R=8 |  |
|  | 6 | 0.77687 | 0.527633 |  |
|  | 2 | 0.393469 | 0.221199 |  |
|  | 0 | 0 | 0 |  |
|  | -5 | -2.49034 | -0.86825 |  |
|  |  |  |  |  |

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|  |  |  |  |  |  |  |  |  |
|  | $ | State1 | State2 | State3 | State4 | EMV | Decision |  |
|  | Decision A | 6 | 2 | 0 | -5 | 0.75 | Yes |  |
|  | Decision B | 0 | 0 | 0 | 0 | 0 |  |  |
|  | Probability | 0.25 | 0.25 | 0.25 | 0.25 |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  | Utiles, R=4 | State1 | State2 | State3 | State4 | E[U] | Decision |  |
|  | Decision A | 0.7768698 | 0.3934693 | 0 | -2.490343 | -0.3300009 |  |  |
|  | Decision B | 0 | 0 | 0 | 0 | 0 | Yes |  |
|  | Probability | 0.25 | 0.25 | 0.25 | 0.25 |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  | Utiles, R=8 | State1 | State2 | State3 | State4 | E[U] | Decision |  |
|  | Decision A | 0.5276334 | 0.2211992 | 0 | -0.868246 | -0.0298533 |  |  |
|  | Decision B | 0 | 0 | 0 | 0 | 0 | Yes |  |
|  | Probability | 0.25 | 0.25 | 0.25 | 0.25 |  |  |  |
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|  | U=Ln(X) |  |
|  |  |  |
|  | U=X^(0.5) |  |
|  |  |  |
|  | U=X/(X+K) , K=constant |  |
|  |  |  |
| .    .   |  |  |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | X= | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | | U-Ln | 0.04 | 0.18 | 0.26 | 0.32 | 0.36 | 0.40 | 0.43 | 0.46 | 0.48 | 0.5 | | U-Sqrt | 0.32 | 0.45 | 0.55 | 0.63 | 0.71 | 0.77 | 0.84 | 0.89 | 0.95 | 1.0 | | U-Exp | 0.18 | 0.33 | 0.45 | 0.55 | 0.63 | 0.70 | 0.75 | 0.80 | 0.83 | 0.86 | | U-K | 0.17 | 0.29 | 0.38 | 0.44 | 0.50 | 0.55 | 0.58 | 0.62 | 0.64 | 0.67 | | K= | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | | R= | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |   . | | |

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|  | **Utility Curves: Empirical Data** |  |

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| **Empirical Data** |
| Estimate a utility curve from observed behavior. A procedure will be presented but mostly the intent of this section is to represent a large class of approaches that will emerge from the logic of utility theory through observing the data. |

Consider a decision maker repeatedly faced with the lottery.

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|  | **A-1** |  |  |  |  | Payoffs |  | Scale Payoffs  X=($+$10)/$50 | Utiles |
|  |  |  |  | (P) | Win | **+$K** |  | XK | U(XK) |
|  |  | GO |  |  |  |  |  |  |  |
|  |  |  |  | (1–P) | Lose | **–$10** |  | 0 | U(0)=0 |
|  |  | NOGO |  |  |  | **$0** |  | 0.2 | U(0.2)=0.5 |
|  |  |  |  |  |  |  |  |  |  |

Since the two fixed values are **–**$10 and $0, arbitrarily set scaling equation, scale values, and utile values as reference values.

Arbitrarily set the scaling equation as X=($+$10)/$50

Then, for payoff –$10, X=(–$10+$10)/$50=0. Arbitrarily set U(0)=0 as a reference utile.

Next, for payoff $0, X=($0+$10)/$50=0.2. Arbitrarily set U(0.2)=0.5 as a reference utile.

Now, let XK=($K+$10)/$50

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| A “GO” decision requires the expected utility of the risky decision to be greater than the utility of the certainty decision. Thus,  P\*U(XK)+(1–P)\*U(0) > U(0.2)  P\*U(XK)+(1–P)\* (0) > (0.5)  U(XK)>0.5/P | A “NOGO” decision requires the expected utility of the risky decision to be less than the utility of the certainty decision. Thus,  P\*U(XK)+(1–P)\*U(0) < U(0.2)  P\*U(XK)+(1–P)\* (0) < (0.5)  U(XK)<0.5/P |

Results from multiple decisions were recorded.

The decision result (GO or NOGO), $K, and P were collected for each decision.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $K= | 1.1 | 1.2 | 2.1 | 2.3 | 4.9 | 5.2 | 8.1 | 8.2 | 10.8 | 15 |
| P= | 0.91 | 0.95 | 0.9 | 0.85 | 0.79 | 0.72 | 0.71 | 0.68 | 0.68 | 0.55 |
| Decision | NOGO | GO | GO | NOGO | GO | NOGO | GO | NOGO | GO | NOGO |

Then, XK and 0.5/P were determined for each decision.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| XK=($K+$10)/$50 | | | | | | | | | | |
| Decision | NOGO | GO | GO | NOGO | GO | NOGO | GO | NOGO | GO | NOGO |
| XK= | 0.222 | 0.224 | 0.242 | 0.246 | 0.298 | 0.304 | 0.362 | 0.364 | 0.416 | 0.5 |
| 0.5/P= | 0.549 | 0.526 | 0.556 | 0.588 | 0.633 | 0.694 | 0.704 | 0.735 | 0.735 | 0.909 |

Fit an exponential utility curve, U(X)=1–exp(X/R), through the bounds of GO and NOGO utility limits.

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The resulting utility curve was found to be U(X)=1–exp(X/R), for R=0.29



